Universal Ratios in the 2-D Tricritical Ising Model

D. Fioravanti[1,2], G. Mussardo[2,3] and P. Simon[1,2]

[1]*International School for Advanced Studies, Via Beirut 2-4, 34014 Trieste, Italy.*
[2]*Instituto Nazionale di Fisica Nucleare, Sezione di Trieste*
[3]*Dipartimento di Fisica, Universita dell'Insubria, Como, Italy*

We consider the universality class of the two–dimensional Tricritical Ising Model. The scaling form of the free–energy naturally leads to the definition of universal ratios of critical amplitudes which may have experimental relevance. We compute these universal ratios by a combined use of results coming from Perturbed Conformal Field Theory, Integrable Quantum Field Theory and numerical methods.

An unifying principle in the study of critical phenomena goes under the name of *universality* [1]. In the vicinity of a phase transition, when the correlation length is much larger than any microscopic scale, one can assign each system to a universality class, which is identified by its dimensionality D, the symmetry properties of the order parameters and the number of relevant fields. The first characteristic of a given universality class is the set of critical exponents, expressed in terms of algebraic expressions of the conformal dimensions of the relevant fields. Additional data of a universality class may be derived by the scaling properties of the free–energy alone. These data – called *universal ratios* – are pure numbers, obtained by taking particular combinations of various thermodynamical amplitudes in such a way to cancel any dependence on the microscopic scales. Together with critical exponents, universal ratios are ideal fingerprints of the universality classes. From an experimental point of view, there is by now a large literature on universal ratio measurements of various systems extending from binary fluids to magnetic systems and polymer conformations (for an extensive review on the subject, see [2]).

In recent years, due to the theoretical progress achieved in the study of two–dimensional models (at criticality by the methods of Conformal Field Theory (CFT) [3], and away from criticality by the approach of Perturbed Conformal Theories [4]), several universal quantities have been computed by different techniques for a large variety of bidimensional systems, such as the self–avoiding walks [5,6], the Ising model [7–10], the q-state Potts model [11], to name few. In this letter we will focus on the first determination of some universal ratios relative to the class of universality of the 2–D Tricritical Ising Model (TIM) for which very few universal quantities are known (see [2,14]). Whereas the 3–D TIM describes, for instance, the universality class of an anti–ferromagnet with strong uniaxial anisotropy like $FeCl_2$, its $2D$ version can describe the tricritical behaviour of a binary mixture of thin films of $He^3 - He^4$ [12] or order–disorder transitions in absorbed systems [13] (for a review on the theory of tricritical points, see [14]). Hence there is an obvious interest in computing the amplest set of universal data for this universality class and in testing the theoretical predictions versus their experimental determinations.

In a continuum version of the TIM (which is, after all, a particular representative of this universality class), it is convenient to adopt a Landau–Ginzburg (LG) formulation based on a scalar field $\Phi(x)$ with Φ^6 interaction. The LG approach permits to have a clear bookkeeping of the symmetry properties of each order parameter and to easily understand the phase diagram of the model, at least qualitatively. The class of universality of the TIM is then described by the LG euclidean action

$$\mathcal{A} = \int d^D x \left[\frac{1}{2}(\partial_\mu \Phi)^2 + g_1\Phi + g_2\Phi^2 + g_3\Phi^3 + g_4\Phi^4 + \Phi^6 \right]$$

with the tricritical point identified by the bare conditions $g_1 = g_2 = g_3 = g_4 = 0$. Adopting a magnetic terminology, the statistical interpretation of the coupling constants is as follows: g_1 plays the role of an external magnetic field h, g_2 measures the displacement of the temperature from its critical value $(T - T_c)$, g_3 may be regarded as a staggered magnetic field h' and finally g_4 may be thought as a chemical potential μ for the vacancy density. Dimensional analysis shows that the upper critical dimension of the model is $D = 3$, where tricritical exponents are expected to have their classical values (apart logarithmic corrections). In two dimensions, although the mean field solution of the model cannot be trusted for the strong fluctuations of the order parameters, an exact solution at criticality is provided by CFT. In fact, the TIM is described by the second model of the unitary minimal series of CFT [3], with central charge equal to $C = \frac{7}{10}$. There are six primary fields, identified with the normal ordered composite LG fields [15], which close an algebra under the Operator Product Expansion (OPE). Only four of them are relevant (*i.e.* with conformal dimension $\Delta < 1$): $\sigma = \varphi_1 \equiv \Phi$ ($\Delta_1 = \frac{3}{80}$), $\varepsilon = \varphi_2 \equiv: \Phi^2 :$ ($\Delta_2 = \frac{1}{10}$), $\sigma' = \varphi_3 \equiv: \Phi^3 :$ ($\Delta_3 = \frac{7}{16}$) and $t = \varphi_4 \equiv: \Phi^4 :$ ($\Delta_4 = \frac{3}{5}$). The fields ε and t are even under the Z_2 spin–symmetry whereas σ and σ' are odd. There is another Z_2 symmetry of the model (related to its self-duality), under which $D\varepsilon D^{-1} = -\varepsilon$, $DtD^{-1} = t$, whereas the magnetic order parameters are mapped onto their corresponding disorder parameters. Each of the above relevant fields can be used to move the TIM away from criticality (the

resulting phases of the model are discussed in [16]).

In order to derive the scaling form of the free energy and the set of universal ratios for the 2–D TIM, let us first normalise the two-point functions of the fields as $\langle \varphi_i(r)\varphi_i(0)\rangle \sim \frac{A_i}{r^{4\Delta_i}}$ when $r \to 0$ (in the perturbed CFT approach to the model, $A_i = 1$). When the TIM is moved away from criticality by means of one (or several) of its relevant fields, with the resulting action $\mathcal{A} = \mathcal{A}_{CFT} + \sum_p g_p \int d^2x\, \varphi_p(x)$, a finite correlation length ξ generally appears. Its scaling form may be written in four possible equivalent ways, according to which coupling constant is selected out as a prefactor

$$\xi = a\,(K_i g_i)^{-\frac{1}{2-2\Delta_i}}\,\mathcal{L}_i\left(\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}\right)\,, \tag{1}$$

where a is some microscopic length scale, $\phi_{ji} \equiv \frac{1-\Delta_j}{1-\Delta_i}$, and $\mathcal{L}_i$ are universal homogeneous scaling functions of the ratios $\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}$. The terms K_i are non–universal metric factors which depend on the unit chosen for measuring the external source g_i, alias on the particular realization of the universality class. Let $f[g_1, g_2, g_3, g_4]$ be the singular part of the free–energy (per unit volume). According to which coupling constant is selected out as a prefactor, it can be parameterised in four possible equivalent ways as:

$$f[g_1, g_2, g_3, g_4] \equiv (K_i g_i)^{-\frac{1}{1-\Delta_i}}\,\mathcal{F}_i\left(\frac{K_j g_j}{(K_i g_i)^{\phi_{ji}}}\right), \tag{2}$$

where $\mathcal{F}_i$ are scaling functions. For the Vacuum Expectation Value (VEV) of the fields φ_j in the ith direction (i.e. for the off–critical theory finally obtained by $g_i \neq 0$, $g_k = 0$, $k \neq i$), we have

$$\langle \varphi_j \rangle_i = -\frac{\partial f}{\partial g_j}\bigg|_{g_k=0} \equiv B_{ji}\, g_i^{\frac{\Delta_j}{1-\Delta_i}}\,, \tag{3}$$

where, from (2), $B_{ji} \sim K_j K_i^{\frac{\Delta_j}{1-\Delta_i}}$. In a similar manner, for the generalized susceptibilities we have

$$\hat{\Gamma}_{jl}^i = -\frac{\partial^2 f}{\partial g_l \partial g_j}\bigg|_{g_k=0} \equiv \Gamma_{jl}^i\, g_i^{\frac{\Delta_j+\Delta_l-1}{1-\Delta_i}}\,, \tag{4}$$

where, from (2), $\Gamma_{jl}^i \sim K_j K_l K_i^{\frac{\Delta_j+\Delta_l-1}{1-\Delta_i}}$. These quantities are obviously symmetric in the lower indices ($\hat{\Gamma}_{22}^i$ and $\hat{\Gamma}_{11}^i$ are respectively the usual specific heat and magnetic susceptibility in the ith direction). Similarly, for the correlation length we have $\xi_i = a\,\xi_0\, g_i^{-\frac{1}{2-2\Delta_i}}$, with $\xi_0 \sim K_i^{-\frac{1}{2-2\Delta_i}}$. From the above formulas, appropriate combinations can be found such that the non–universal metric factors K_i cancel out. Some of the 2–D universal ratios are:

$$(R_c)_{jk}^i = \frac{\Gamma_{ii}^i \Gamma_{jk}^i}{B_{ji} B_{ki}} \tag{5}$$

$$(R_\chi)_j^i = \Gamma_{jj}^i B_{jj}^{\frac{\Delta_j-1}{\Delta_j}} B_{ji}^{\frac{1-2\Delta_j}{\Delta_j}} \tag{6}$$

$$R_\xi^i = \left(\Gamma_{ii}^i\right)^{1/2} \xi_i^0 \tag{7}$$

$$(R_A)_j^i = \Gamma_{jj}^i B_{ii}^{\frac{-2\Delta_j-\Delta_i+2}{\Delta_i}} B_{ij}^{\frac{2\Delta_j-2}{\Delta_i}} \tag{8}$$

$$(Q_2)_{jk}^i = \frac{\Gamma_{jj}^i}{\Gamma_{jj}^k}\left(\frac{\xi_k^0}{\xi_j^0}\right)^{2-4\Delta_j}. \tag{9}$$

In this letter we only consider the case $i = 1, 2$, which correspond to the most important physical deformations of the model (the magnetic and the thermal ones), i.e. those which are most accessible from an experimental point of view. For both the magnetic and thermal deformation there are no mixing among the conformal fields due to ultraviolet renormalization [17,18]. A complete analysis relative to all deformations of the TIM and the theoretical details of our approach will be published elsewhere [21].

The ϵ perturbation around the critical TIM is integrable and its behavior is governed by the E_7 algebra [19]. Therefore the B_{j2}'s in eq. (3) have been computed exactly in [20]. On the other hand, the σ perturbation is non-integrable (numerical indications were discussed in [16]). In this case, the B_{j1}'s have been numerically evaluated in [22] by using the so–called Truncated Conformal Space Approach (TCSA) [23]. This method consists in diagonalizing the off–critical Hamiltonian on a cylinder in a truncated conformal basis of the critical TIM such that an extimation of $\langle \varphi_j \rangle_1$ can be obtained from the knowledge of the eigenvectors (only the ground state eigenvector is needed for the VEV). All these calculations can be easily performed by means of the numerical program of ref. [24].

In order to estimate the universal ratios, it is still necessary to calculate the Γ_{jk}^i's. Their values can be extracted in two different ways. The first method is purely numerical and of immediate use, since it consists in employing the TCSA to compute numerically the derivative $\frac{\partial}{\partial g_k}\langle \varphi_j \rangle_i$ (details will be found in [21]). The second method is based on the fluctuation–dissipation theorem which permits to express the generalised susceptibilities as

$$\hat{\Gamma}_{jk}^i = \int d^2x\, \langle \varphi_j(x)\varphi_k \rangle_i^c\,, \tag{10}$$

where $\langle \cdots \rangle^c$ indicates the connected correlator. Therefore in this second approach we first need to evaluate the 2−point correlation functions and then to perform the integration. For our calculation of the universal ratios, we have employed both methods, finding an agreement in their final outputs. Let us briefly discuss the second method. First of all, write the integral (10) in polar coordinate as $\hat{\Gamma}_{jk}^i = 2\pi \int dr\, r\, \langle \varphi_j(r)\varphi_k \rangle_i^c$. Secondly, decompose the integral over r into two integrals over the regions $0 < r < R$ and $r \geq R$ with $R \sim \xi$. When $r < R$, the correlation function $\langle \varphi_j(r)\varphi_k(0)\rangle_i$ can be efficiently evaluated by using a short–distance expansion [17]

$$\langle \varphi_j(r)\varphi_k(0)\rangle_i = \sum_l C^l_{jk}(r)\langle\varphi_l\rangle_i \qquad (11)$$

where the non–analytic dependence on the coupling constant is completly encoded into the VEV's, whereas the structure constants $C^l_{jk}(r)$ can be evaluated perturbatively in g

$$C^l_{jk}(r) = r^{2(\Delta_l - \Delta_j - \Delta_k)} \sum_{n=0}^{\infty} C^{l(n)}_{jk} \left(g_i r^{2-2\Delta_i}\right)^n \quad . \quad (12)$$

For the TIM, $C^{l(0)}_{jk}$ have been computed in [16] whereas their first correction can be obtained by the formula

$$C^{l(1)}_{jk} = -\int{}' d^2z \, \langle\varphi_l(\infty)\varphi_i(z)\varphi_k(1)\varphi_j(0)\rangle_{CFT} \quad , \quad (13)$$

where the prime indicates a suitable infrared regularization of the integral. As shown in [25], an efficient way to compute the regularised integrals is through a Mellin transformation. Hence, the calculation of the above integral (13) on the conformal functions plus the knowledge of the various expectation values $\langle\varphi_l\rangle_i$ enables us to reach a quite accurate approximation of $\langle\varphi_j(r)\varphi_k(0)\rangle_i$ in the ultraviolet limit, *i.e.* for $r < R$. By choosing $R \sim \xi$, one can obtain an overlap between the ultraviolet and the infrared representations of the correlation functions. The latter is expressed by means of the spectral series of the correlators on the massive states $\mid A_k(\theta)\rangle$ of the off–critical theory

$$\langle\varphi_j(x)\varphi_k(0)\rangle_c = \sum_{n=1}^{\infty} g_n(r) \quad , \qquad (14)$$

where

$$g_n(r) = \int \frac{d\theta_1}{2\pi} \cdots \frac{d\theta_n}{2\pi} \langle 0|\varphi_j(0)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle \times$$
$$\langle A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)|\varphi_k(0)|0\rangle \, e^{-r\sum_{k=1}^{n} m_k \cosh\theta_k} \quad .$$

As tested in several examples (see, for instance [6,7,17,25,26]), the above series (14) converges very fast even for $r \sim \xi$ so that its truncation to the lowest terms is able to capture the correct behaviour of the correlator in the interval $r \geq \xi$. For the integrable theory defined by the thermal deformation of the TIM, one can truncated the series up to the lowest 2–particle states, with the relative matrix elements computed along the lines of the refs. [7,17,27,28]. For the non–integrable theory defined by the magnetic deformation of the TIM, it is hard to go beyond the one–particle matrix elements and one has to be satisfied with the estimate of the correlators obtained by the one–particle contributions only: since this theory has two lowest masses with mass ratio $m_2 \simeq 2m_1 \cos\frac{\pi}{5}$ [16], in this case we have

$$\langle\varphi_j(r)\varphi_k(0)\rangle_i \approx \frac{1}{\pi}\left(f^1_j f^1_k K_0(m_1 r) + f^2_j f^2_k K_0(m_2 r)\right)$$

where $K_0(x)$ is the modified Bessel function and the indices $1, 2$ refer to the first and second massive states. The one–particle matrix elements of this model $f^k_j = \langle 0|\varphi_j(0)|A_k\rangle$ can be also computed numerically by using the TCSA [21].

Once an overlap of the short and large distance expansions of the correlators in the region $r \sim \xi$ has been checked, a numerical integration of the correlators provides the Γ^i_{jk}'s. An explicit test of the validity of the above method (with a corresponding estimate of its errors) is provided by the comparison of the values of Γ^i_{ik} (obtained by the numerical integration) with their exact determination extracted by the Δ−theorem sum rule, when this theorem applies [18]:

$$\Gamma^i_{ik} = -\frac{\Delta_k}{1 - \Delta_k} B_{ki} \quad . \qquad (15)$$

This check shows that the uncertanties for Γ^i_{jk} is at worst about 5%, better for the strongest relevant operators. Gathering all these results, a set of universal ratios for the TIM have been obtained. Some of them are exact, like $(R_c)^1_{1,k} = \frac{240}{5929}\Delta_k$, $(R_c)^2_{2,k} = \frac{10}{81}\Delta_k$ ($k = 1,\ldots,4$). We have also computed those relative to the low and high temperature phase of the model (Table 1). An interesting universal ratio is provided in this case by the correlation length prefactors $\xi^{\mp}_0$, below and above the critical temperature (as extracted from the correlation function of the magnetic operator using its duality properties)

$$\frac{\xi^-_0}{\xi^+_0} = 2\cos\left(\frac{5\pi}{18}\right) \approx 1.28557... \qquad (16)$$

which can be inferred by the exact mass spectrum of the model and the parity properties of the excitations [16,19].

In summary, we have combined techniques coming from CFT, integrable models and numerical methods to obtain for the first time a set of universal quantities for the class of universality of the $2D$ Tricritical Ising Model. It would be interesting to have an experimental determination of these quantities and a comparison with the theoretical predictions presented here.

[1] M.E. Fisher, *Phys. Rev. Lett.* **16** (1966), 11; R.B. Griffith, *Phys. Rev. Lett.* **24** (1970), 1479.

[2] V. Privman, P.C. Hohenberg and A. Aharony, *Universal Critical–Point Amplitude Relations*, in Phase Transitions, vol. 14, eds. Domb and Green.

[3] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl.Phys.* **B241** (1984), 333; D. Friedan, Z. Qiu and S. Shenker, *Phys.Rev.Lett.* **52** (1984), 1575.

[4] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* **48** (1987), 1090; *Int. Journ. Mod. Phys.* **A3** (1988), 743.

[5] J.L. Cardy, H. Saleur *J. Phys.* **A 22** (1989), L601; P. Fendley, H. Saleur *Nucl. Phys.* **B 388** (1992), 609; Al.B. Zamolodchikov, *Nucl. Phys.* **B 432** (1994), 427; J. Cardy, *Phys. Rev. Lett.* **72** (1994), 1580.

[6] J.L. Cardy and G. Mussardo, *Nucl. Phys.* **B 410** (1993), 451.

[7] G. Delfino and G. Mussardo, *Nucl. Phys.* **B 455** (1995), 724; G. Delfino and P. Simonetti, *Phys. Lett.* **B 383** (1996), 450.

[8] G. Delfino, *Phys. Lett.* **B 419** (1998), 291.

[9] J. Salas and A. D. Sokal, cond-mat/99041038

[10] M. Caselle, M. Hasenbusch, hep-th/9911216; M. Caselle, P. Grinza, N. Magnoli, hep-th/9909065

[11] G. Delfino and J. Cardy, *Nucl. Phys.* **B519** (1998) 551; G. Delfino, G. T. Barkema and J. Cardy, cond-mat/9908453; G.Delfino and J. Cardy hep-th/0002122; M. Caselle, R. Tateo, S. Vinti, *Nucl. Phys.* **B 562** (1999), 549.

[12] A.N. Berker and D.R. Nelson, *Phys. Rev.* **B 19** (1979), 2488.

[13] M.J. Tejwani, O. Ferreira, O.E. Vilches, *Phys.Rev.Lett.* **44** (1980), 152; W. Kinzel, M. Schick and A.N. Berker, in *Ordering in Two Dimensions*, Elsevier North Holland (1980), 381.

[14] I.D. Lawrie, S. Sarbach, in *Phase Transitions and Critical Phenomena*, vol. 9, C. Domb and J. Lebowitz (Eds.), New York (Academic Press 1984).

[15] A.B. Zamolodchikov, *Sov.Journ.Nucl.Phys.* **44** (1986), 529.

[16] M. Lassig, G. Mussardo and J.L. Cardy, *Nucl. Phys.* **B 348** (1991), 591.

[17] Al.B. Zamolodchikov, *Nucl. Phys.* **B 348** (1991), 619.

[18] G. Delfino, P. Simonetti and J.L. Cardy, *Phys. Lett.* **B 387** (1996), 327.

[19] P. Christe, G. Mussardo, *Nucl. Phys.* **B330** (1990), 465; V.A. Fateev, A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A5** (1990), 1025.

[20] V.A. Fateev, S. Lukyanov, A.B. Zamolodchikov and Al.B. Zamolodchikov,*Nucl. Phys.* **B 516** (1998), 652.

[21] D. Fioravanti, G. Mussardo and P. Simon, in preparation.

[22] R. Guida and N. Magnoli, *Phys. Lett.* **B411** (1997), 127.

[23] V.P. Yurov and Al.B. Zamolodchikov, *Int. J. Mod. Phys.* **A 6** (1991), 4557.

[24] M. Lässig and G. Mussardo, *Comp. Phys. Comm.* **66** (1991) 71.

[25] R. Guida and N. Magnoli, *Int. J. Mod. Phys.* **A13** (1998) 1145.

[26] G. Delfino and G. Mussardo, *Phys. Lett.* **B 324** (1994), 40.

[27] C. Acerbi, G. Mussardo and A. Valleriani, *Int. J. Mod. Phys.* **A11** (1996) 5327.

[28] M. Karowski, P. Weisz, *Nucl. Phys.* **B139** (1978), 445; F.A. Smirnov, Form Factors in Completely Integrable Models of Quantum Field Theory (World Scientific), 1992 and references therein.

Table 1: Amplitude ratios $R^2_{jk} = \frac{\Gamma^{2+}_{jk}}{\Gamma^{2-}_{jk}}$.

$R^2_{11} = 3.54$	$R^2_{13} = -2.06$
$R^2_{22} = 1$	$R^2_{24} = -1$
$R^2_{33} = 1.30$	$R^2_{44} = 1$

Table 2: Universal ratios $(R_c)^1_{jk}$ and $(R_c)^{2-}_{jk}$.

$(R_c)^1_{22} = 1.05 \ 10^{-2}$	$(R_c)^1_{23} = 4.85 \ 10^{-2}$
$(R_c)^1_{24} = 6.7 \ 10^{-2}$	$(R_c)^1_{33} = 3.8 \ 10^{-1}$
$(R_c)^{2-}_{11} = 2.0 \ 10^{-3}$	$(R_c)^{2-}_{14} = -2.34 \ 10^{-2}$
$(R_c)^{2-}_{13} = 1.79 \ 10^{-2}$	$(R_c)^{2-}_{33} = 3.4 \ 10^{-1}$

Table 3: Universal ratio $(R_\chi)^i_j$ for $i, j = 1, 2$.

$(R_\chi)^1_1 = 3.897 \ 10^{-2}$	$(R_\chi)^{2+}_2 = 0.1111$
$(R_\chi)^1_2 = 0.116$	$(R_\chi)^{2-}_1 = 0.040$
$(R_\chi)^{2+}_1 = 0$	$(R_\chi)^{2-}_2 = 0.1111$

Table 4: Universal ratios R^i_ξ and $(R_A)^i_j$ for $i, j = 1, 2^-, 2^+$.

$R^1_\xi = 7.557 \ 10^{-2}$	
$R^{2+}_\xi = 1.0784 \ 10^{-1}$	$R^{2-}_\xi = 8.389 \ 10^{-1}$
$(R_A)^1_{2+} = 0$	$(R_A)^1_{2-} = 3.918 \ 10^{-2}$
$(R_A)^{2+}_1 = 2.958 \ 10^{-1}$	$(R_A)^{2-}_1 = 8.260 \ 10^{-1}$

Table 5: Universal ratios $(Q_2)^i_{jk}$ for $i, j, k = 1, 2^+, 2^-$.

$(Q_2)^1_{2+1} = 1.260$	$(Q_2)^1_{2-1} = 1.884$
$(Q_2)^1_{2+2+} = 1.973$	$(Q_2)^1_{2+2-} = 1.320$
$(Q_2)^{2+}_{11} = 1.56$	$(Q_2)^{2-}_{11} = 0.442$
$(Q_2)^{2+}_{12-} = 1.70$	